Chapter 1
On the Relevance of Theoretical Results to Voting System Choice

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Abstract Some thirty years ago Fishburn, Niemi, Richelson, Riker and Straffin published first systematic comparisons of voting procedures using several criteria of systems performance. Still today Richelson's set of criteria – and his list of voting systems – is perhaps the most extensive in the literature. Some of the criteria, notably monotonicity and the Condorcet criteria, introduced by these authors still play a central role in the present day treatises on voting systems. This paper discusses the relevance of performance criteria and various paradoxes to the choice of voting systems. We first outline a procedure whereby the criteria could be systematically utilized. This is analogous to a wide class of multi-criterion choice problems. Apart from relatively trivial settings, this procedure leaves many problems open. These, in turn, seem to depend on context-related considerations. Moreover, it is argued that individual preference rankings may not be most appropriate way to approach voting systems: cyclic individual preferences sometimes make perfect sense and often people are capable of much more refined opinion expression than preference ranking.

1.1 Introduction

The first systematic comparisons of voting procedures appeared in the 1970's. The journal *Behavioral Science* became a major forum for these early publications. Especially notable are the article by Peter C. Fishburn (1971) and a series of works by Jeffrey T. Richelson. This series culminated in a summary (Richelson 1979) that is perhaps the most extensive of its kind in terms of both the number of systems and the number of criteria. These were followed by a book-length treatise by Philip D. Straffin, Jr (1980) and perhaps most notably by William H. Riker's (1982) *magnum opus.* While these texts explicitly dealt with voting systems, they were preceded and inspired by several path-breaking works in the more general field of social choice functions, e.g. (Fishburn 1973), (Fishburn 1977) and (Young 1975). The history of voting procedures had also been discussed in Black (1958) and Riker (1961). The wider public was first made aware of the theory of comparative voting systems by an article in *Scientific American* written jointly by Richard G. Niemi and Riker (1976).

From those early years on there has been a relatively clear distinction between theoretical and applied works. Fishburn, Richelson and Young are obviously theoretical scholars, while Riker and Straffin had a more applied focus. Indeed, Riker (1982) can be seen as an attempt to justify a specific theory of democracy by invoking theoretical results achieved in social choice theory. More specifically, Riker argues that since

- all known voting procedures have at least one serious flaw,
- voting equilibria are extremely rare in multidimensional spatial voting models, and
- strategic manipulation opportunities are ubiquitous,

it is erroneous to equate voting results with the "will of the people" or expressions of collective opinion for the reason that the latter is a meaningless notion. Hence, defining democracy as a system ruled in accordance with the will of the people is indefensible. His favorite – liberal – view of democracy, on the other hand, is immune to the negative results of social choice theory because it does not require more of a voting – or, more generally, ruling – system than that it enables the voters to get rid of undesired rulers. For this purpose, continues the argument, the plurality rule is a particularly apt instrument.

Riker's view is thought-provoking. Many authors, while accepting its premises based on social choice theory, have questioned the conclusions (Lagerspetz 2004; Mackie 2003; Nurmi 1984; Nurmi 1987). This paper dwells on the premises and their significance for voting system design. We shall first outline the standard view which looks at various voting systems and evaluates them in terms of criteria of performance. This approach in essence deems all systems satisfying a given criterion of performance as equivalent and those which don't also equivalent. Starting from the 1960's a rich literature on probability and simulation modeling of voting systems performance has emerged to give a somewhat more nuanced picture (Klahr 1966; Niemi & Weisberg 1972). We shall discuss the nature and relevance of these results. We then deal with the intuitive difficulty of devising examples of various criterion violations and discuss whether this should play a role in voting system evaluations. Finally, we shall scrutinize the "givens" of the theory used in the evaluation.

1.2 The standard approach

The motivation for introducing a new voting system or criticizing an old one is often a counterintuitive or unexpected voting outcome. A case in point is Borda's memoir where he criticized the plurality voting and suggested his own method of marks (McLean & Urken 1995). With time this approach focusing on a specific flaw of a system has given way to studies dealing with a multitude of systems and their properties. An example of such studies (e.g. Nurmi 2002, 36; Nurmi 2006, 136-137) is summarized in Table 1.1.

	Criterion							
Voting system	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
Amendment	1	1	1	1	0	0	0	0
Copeland	1	1	1	1	1	0	0	0
Dodgson	1	0	1	0	1	0	0	0
Max-min	1	0	1	1	1	0	0	0
Kemeny	1	1	1	1	1	0	0	0
Plurality	0	0	1	1	1	1	0	1
Borda	0	1	0	1	1	1	0	1
Approval	0	0	0	1	0	1	0	1
Black	1	1	1	1	1	0	0	0
Pl. runoff	0	1	1	0	1	0	0	0
Nanson	1	1	1	0	1	0	0	0
Alternative vote	0	1	1	0	1	0	0	0

Table 1.1 A Comparison of voting procedures

Here criterion *a* denotes the Condorcet winner criterion, *b* the Condorcet loser one, *c* strong Condorcet criterion, *d* monotonicity, *e* Pareto, *f* consistency, *g* independence of irrelevant alternatives and *h* invulnerability to the no-show paradox. A "1" ("0", respectively) in the table means that the system represented by the row satisfies (violates) the criterion represented by the column.

The systems are viewed as choice rather than preference functions. This distinction makes a difference especially in the case of the Kemeny rule. As a preference function it is consistent (Young & Levenglick 1978), but as a choice rule it isn't.[1] It will be recalled that choice functions map preference profiles into subsets of alternatives. Denoting by Φ the set of all preference profiles and by A the set of alternatives, we thus have

[1] I am grateful to Dan Felsenthal for calling my attention to the apparent discrepancy between Young and Levenglick's claim that the Kemeny rule satisfies both the Condorcet winner criterion and consistency, and Fishburn's demonstration that the rule is not consistent.

$$f : \Phi \rightarrow 2^A$$

for social choice functions.

Preference functions, in contradistinction, map preference profiles into rankings over alternatives (cf. social welfare functions). I.e.

$$F : \Phi \rightarrow \mathcal{R}$$

where $\mathcal{R}$ denotes the set of all preference rankings over A.

Consider now a partition of a set N of individuals with preference profile ϕ into two separate sets of individuals N_1 and N_2 with corresponding profiles ϕ_1 and ϕ_2 over A and assume that $f(\phi_1 \cap \phi_2) \neq \emptyset$. The social choice function f is consistent iff $f(\phi_1 \cap \phi_2) = f(\phi)$, for all partitionings of the set of individuals.

The same definition can be applied to social preference functions. F is consistent iff whenever $F(\phi_1) \cap F(\phi_2) \neq \emptyset$ implies that $F(\phi_1) \cap F(\phi_2) = F(\phi)$.

It turns out that, like all Condorcet extensions, Kemeny's rule is an inconsistent social choice function. An example is provided by Fishburn (1977, 484). However, as a preference function it is consistent, i.e. whenever two distinct subsets of individuals come up with some common preference rankings, these common rankings must also be the result when the sub-profiles are put together. Young's result that all Condorcet extensions are inconsistent is visible in Table 1.1 where all those systems with a 1 in column a have a 0 in column *f*. The satisfaction of the Condorcet winner criterion is, however, just a sufficient, not necessary, condition for inconsistency: plurality runoff and Hare's system fail on both the Condorcet winner criterion and on consistency.

Of particular interest in Table 1.1 is column h, the invulnerability to the no-show paradox. One of the main motivations for elections is to get an idea of voter preferences. Systems that are vulnerable to the no-show paradox are at least prima facie incompatible with this motivation. It has been shown by Moulin (1988) and Pérez (1995) that all Condorcet extensions are vulnerable to the no-show paradox and, indeed, as shown by Pérez (2001), most of them to the strong version thereof whereby by abstaining a group of voters may get their first-ranked alternative elected, while some other alternatives would be elected if they would vote according to their preferences.

At first sight, monotonicity is closely related to invulnerability to the no-show paradox. On closer scrutiny the situation gets more nuanced. Firstly, among monotonic systems there are both systems that are vulnerable to the no-show paradox and those that are not (see e.g. Nurmi 2002, 103). In other words, monotonicity does not imply invulnerability to the no-show paradox. By Moulin's result all monotonic Condorcet extensions – e.g. Copeland's and Kemeny's methods – are vulnerable to the no-show paradox. More obviously, monotonicity does not imply vulnerability either since e.g. the plurality rule is both monotonic and invulnerable to the paradox. The same is true of the Borda count. But what about non-monotonicity? Does it imply vulnerability? Again Moulin's result instructs us that non-monotonic Condorcet extensions – e.g. Dodgson's and Nanson's methods – are vulnerable. So are plurality runoff and alternative vote. Indeed, in Table 1.1 all non-monotonic systems are vulnerable to the no-show paradox. Campbell and Kelly (2002) have shown, however, that this is not the case in general, i.e. there are non-monotonic systems that are invulnerable to the no-show paradox. These are, however, either non-anonymous or non-neutral (or both). Hence, within the class of anonymous and neutral procedures we get the following table (Table 1.2).

	monotonic	non-monotonic
vulnerable	Copeland	plurality runoff
invulnerable	Borda	empty

Table 1.2 Monotonicity and vulnerability to no-show paradox among anonymous and neutral systems: examples

1.3 Standard approach and system choice

Table 1.1 gives a summary information of some criteria and systems. To justify a "1" in the table one has to show that the criterion represented by the column is under no profile violated by the system represented by the row. To justify a "0" requires no more than an example where the system violates the criterion. This information may be useful in choosing a voting system. Suppose that one is primarily interested in only one criterion, say Condorcet winning. Then one's favorite systems are those with a "1" in column a. This in itself sensible way of proceeding leaves, however, one with many systems. So, we need additional considerations to narrow the choice down.

A more "graded" approach to comparing two systems with respect to one criterion has also been suggested (Nurmi 1991; see also Lagerspetz 2004). The superiority of system A with respect to system B takes on degrees from strongest to weakest as follows:

1. A satisfies the criterion, while B doesn't, i.e. there are profiles where B violates the criterion, but such profiles do not exist for A.
2. in every profile where A violates the criterion, also B does, but not vice versa.
3. in *practically all profiles* where A violates the criterion, also B does, but not vice versa ("A dominates B almost everywhere").
4. in a plausible probability model B violates the criterion with higher probability than A.
5. in those political cultures that we are interested in, B violates the criterion with higher frequency than A.

We shall return to items 4 and 5 in the next section. Comparing systems with respect to just one criterion is, however, not plausible since criteria tend to be contested not only among the practitioners devising voting systems, but also within the scholarly community. Suppose instead that one takes a more holistic view of Table 1.1 and gives some consideration to all criteria. A binary relation of dominance could then be defined as follows:

Definition 1. A system A (strictly) dominates system B in terms of a set of criteria, if and only if whenever B satisfies a criterion, so does A, but not the other way around.[2]

In Table 1.1 e.g. Kemeny's rule dominates all other systems except Copeland, Black, plurality, Borda and approval voting. Regardless of what relative weights one assigns to various criteria, it seems natural to focus on the undominated systems. Thus in Table 1.1 one is left with the six systems just mentioned.

But all criteria are not of equal importance. Nor are they unrelated. To wit, if a system always ends up with the Condorcet winner, i.e. satisfies criterion a, it also elects the strong Condorcet winner, that is, satisfies criterion c. It is also known that the Condorcet winner criterion is incompatible with consistency (Young 1974a, Young 1974b). Some criteria seem to be context-related in the sense that they lose their practical relevance in some specific contexts. E.g. one could argue that consistency has no practical bearing on committee decisions since the results are always determined by counting the entire set of ballots. This observation notwithstanding, there is a more subtle argument one can build against the standard approach: the finding that a criterion is not satisfied by a system tells us very little – in fact nothing – about the likelihood of violation. For that we need to focus on the likelihood of "problematic" profiles since these – together with choice rules – determine the outcomes. It is when we compare the outcomes with the profiles that we find out whether a criterion violation has occurred.

1.4 How often are criteria violated?

To find out how often a given system violates a criterion – say, elects a Condorcet loser – one has to know how often various preference profiles occur and how these are mapped into voting strategies by voters. Once we know these two things, we can apply the system to the voting strategy n-tuples (if the number of voters is n), determine the outcomes, and, finally, compare these with the preference profile to find out whether the choices dictated by

[2] A referee suggests a more general version: "To whatever degree B satisfies a criterion, A satisfies it to at least the same degree, but not the other way around." Since we are primarily dealing with dichotomous criteria, we shall use the less general version.

the criterion contradict those resulting from the profile, e.g. if an eventual Condorcet loser was chosen. Traditionally, two methods have been resorted to in estimating the frequency of criterion violations: (i) probability modeling, and (ii) computer simulations. Both are based on generating artificial electorates and calculating how frequently the criterion is violated or some other incompatibility encountered in these electorates.

The literature on probability and computer simulations is vast (see e.g. Gehrlein 1997; Gehrlein 2002; Gehrlein 2006; Gehrlein & Lepelley 2004; Lepelley 1993; Merlin et al. 2000; Saari & Tataru 1999). Of particular interest has been the occurrence of cyclic majorities. The early models were based on the impartial culture (IC) assumption. Under it each voter is randomly and independently assigned to a preference ranking over alternatives. So, the voters are treated as random samples – with replacement – from a uniform distribution over all preference rankings. The method devised by Gehrlein and Fishburn (1978) is useful in deriving limit probabilities when the number of voters increases. IC is a variation of the principle of insufficient reason: since we cannot know which preference profiles will emerge in the future, we assume that all individual preference relations are purely random in the sense that each individual's preference relation is independently drawn from a uniform distribution of preferences over all possible preference rankings. Like all versions of the principle of insufficient reason, IC is based on untenable epistemology: it is not possible to derive knowledge about probabilities of rankings from complete ignorance regarding those probabilities. Despite its implausibility, this assumption could still be made because of its technical expediency if one could point out that the results based on IC do not deviate very much from those obtained under other more plausible assumptions. But, alas, this is not the case: the IC simulations results often differ dramatically from other simulation results (see e.g. Nurmi 1999).

Regenwetter et al. (2006) strongly criticize the IC assumption by arguing that it in fact maximizes the probability of majority cycles. Their criticism aims at playing down the empirical significance of the results that – under the IC assumption – suggest that the probability of majority cycles is reasonably high even in the case of just three alternatives. Surely, the fact that the probability of cycles is estimated at 0.09 in IC's when the number of voters approaches infinity and the number of alternatives is 3, does not imply that the probability of cycles would be of the same order in current, past or future electorates. What those results literally state is that if the opinions of the voters resemble IC, then the probability of encountering majority cycles is as specified. The interest of these estimates is not in their predictive success in real world, but in their ability to provide information about variables and parameters that increase or decrease the likelihood of cycles. Probability models are in general more useful in providing this kind of information. Often the interest is not so much in the probability estimates themselves but on their variability under various transformations in the models.

Consider the studies on Condorcet efficiency of various voting procedures, i.e. on the probability that the Condorcet winner is chosen by a procedure under various cultures. Those studies that focus on Condorcet efficiency are typically reporting the probability of the Condorcet winner being chosen, provided that such a winner exists in the profile. In other words, these studies (e.g. Merrill 1984) do not aim at predicting how often Condorcet winners are elected, but, by focusing instead on just those profiles where a Condorcet winner exists, help to identify the propensity of various procedures to elect the Condorcet winner (see also Merrill 1988). Similarly studies reporting the probability of various systems to come up with Condorcet losers are not predicting the relative frequency of Condorcet losers being elected in current elections, but are aiming at disclosing factors, variables or parameters that increase or decrease such choices under profiles where a Condorcet loser exists. Yet, the argument of Regenwetter et al. is supported by simulations where IC assumption is slightly perturbed by assuming that a small minority of the electorate – say, 5 or 10 per cent of the total – forms a homogeneous sub-culture of voters with identical preferences while the rest of the electorate remains an IC. It then turns out that the Condorcet efficiencies of various systems change quite significantly. More importantly, even the ranking of systems in terms of Condorcet efficiency can change for some combinations of alternatives and voters (Nurmi 1992). Similar observation can be made about differences in choice sets of various systems under IC and small perturbations thereof. IC seems to be associated with larger discrepancies of systems than systems where a minuscule group representing identical preferences is immersed in IC (Nurmi 1988; Nurmi 1992).

Despite its tendency to exaggerate Condorcet cycles and dampen Condorcet efficiencies of systems that are not Condorcet extensions, IC may be a useful construct in illuminating the differences of voting rules. By estimating the likelihood that two rules make overlapping choices in IC's we get a profile-neutral view of how far apart they are as choice intuitions. For example, IC simulations suggest – unsurprisingly – that two Condorcet extensions, Copeland's rule and max-min method (also known as Simpson's method), are relatively close to each other in the sense of resulting rarely in distinct choice sets. More interesting is the finding that the Borda count is nearly as close to Copeland's rule as the max-min method is (Nurmi 1988). This is consistent with the relatively high Condorcet efficiency of the Borda count reported in several studies (e.g. Merrill 1984; Nurmi 1988). As is well-known, the Borda scores of alternatives can be computed from the outranking matrix by taking row sums. This binary implementation of the Borda count already hints that, despite its positional nature, the method is reasonably close to the idea that the winners be detected through binary comparisons.

The criticism of IC has so far not produced many alternative culture assumptions. Perhaps the most widespread among the alternative assumptions is that of impartial anonymous culture (IAC). Consider an electorate of n voters considering the set of k alternatives. The number of rankings of alternatives is then $k!$. Let n_i denote the number of voters with i'th preference ranking ($i = 1, \ldots, k!$). Each anonymous profile can be represented by listing the n_i's. The profile satisfies anonymity since transferring j voters from n_s to n_t when accompanied with transferring j voters from n_t to n_s leaves the distribution of voters over preference ranking unchanged. In IAC's every distribution of voters over preference rankings is assumed to be equally probable. This changes the Condorcet efficiency as well as Borda paradox estimates by increasing the former and decreasing the latter (Gehrlein 1997; Gehrlein 2002).

Is IAC then more realistic than IC? Both IC and IAC are poor proxies of political electorates. Given any election result it is inconceivable that the profile emerging in the next election would, with equal probability, be any distribution of voters over preference rankings. The same is true of committees and other bodies making several consecutive collective choices. There is in general far more interdependence between voters than suggested by IAC. Indeed, it can argued (Nurmi 1988a) that in reconstructing the profile transformation over time, one should distinguish two mechanisms: (i) one that determines the initial profile, and (ii) one which determines the changes from one time instant (ballot) to the next. Both IC and IAC collapse these two into one mechanism that generates each voting situation de novo. This is certainly not the way in which everyday experience suggests that opinion distributions are formed. If it were, the electoral campaigns would take on heretofore unknown forms: the distinctions between core constituencies and moving voters would vanish as would that between government and opposition etc. So, it seems that everyday observations fly in the face of IC, IAC and many other models used in generating voter profiles. This does not play down the importance of those models as theoretical tools, i.e. in enhancing our understanding of the mechanisms increasing or decreasing the occurrence of various paradoxes, incompatibilities or discrepancies related to voting systems. Nevertheless, to render choice theoretic results more relevant for the evaluation of voting rules, one should bring the incompatibility results closer to practice by finding out what the problematic profiles look like, i.e. what kinds of opinion distributions underly them. If it is very difficult to envision how those profiles would emerge in practice, then arguably the corresponding incompatibility results do not have much practical importance.

1.5 Counterexamples are sometimes difficult to come by

Summaries like Table 1.1 provide information that is of somewhat asymmetric nature. To prove that a system is incompatible with a criterion one needs to find a profile where – under the assumed mechanism concerning voting behavior – the system leads to a choice that is not consistent with the range of choices allowed for by the criterion. To find such a profile when, theoretically, one should exist, is, however, not always easy. At the behest of and in cooperation with Dan S. Felsenthal the present author embarked upon looking for examples illustrating the incompatibility of the Condorcet winning criterion and invulnerability to the no-show, truncation and twin paradoxes. The background of this search is the result proven by Moulin (1988) and subsequently strengthened by Pérez (2001) saying that all Condorcet extensions are vulnerable to the no-show paradox. In the subsections that follow these incompatibilities are illustrated for some well-known Condorcet extensions (for fuller discussion, see Felsenthal 2010).

1.5.1 Black's procedure

Black' procedure is vulnerable to the no-show paradox, indeed, to the strong version thereof. This is illustrated in Table 1.3.

1 voter	1 voter	1 voter	1 voter	1 voter
D	E	C	D	E
E	A	D	E	B
A	C	E	B	A
B	B	A	C	D
C	D	B	A	C

Table 1.3 Black's system is vulnerable to strong no-show paradox

Here D is the Condorcet winner and, hence, is elected by Black.

Suppose now that the right-most voter abstains. Then the Condorcet winner disappears and E emerges as the Borda winner. It is thus elected by Black. E is the first-ranked alternative of the abstainer. Hence we have a strong version of the paradox.

Truncation paradox is closely related to the no-show one. It occurs whenever a group of individuals gets a better outcome by revealing only part of their preference ranking rather than their full ranking. Obviously, not voting at all is an extreme version of truncation and thus the above can be used to show that Black is also vulnerable to truncation. If more specific demonstration is needed, then one might consider the modification of the above example whereby the right-most voter truncates his preference after A, i.e. does not express any view regarding C and D. Then, the Condorcet winner again disappears and the Borda winner E emerges as the Black winner. Again the strong version of the truncation paradox emerges.

1.5.2 Nanson's method

Nanson's Borda-elimination procedure is vulnerable to the strong version of no-show paradox as well as Table 1.4 illustrates.[3] Here Nanson's method results in B.

5 voters	5 voters	6 voters	1 voter	2 voters
A	B	C	C	C
B	C	A	B	B
D	D	D	A	D
C	A	B	D	A

Table 1.4 Nanson's method is vulnerable to strong no-show paradox

If one of the right-most two voters abstain, C – their favorite – wins. Again the strong version of no-show paradox appears.

The twin paradox occurs whenever a voter is better off if one or several individuals, with identical preferences to those of the voter, abstain. In Table 1.4 we have an instance of the twin paradox as well: if there is only one CBDA voter, C wins. If he is joined by another, B wins.

Nanson is also vulnerable to truncation: if the 2 right-most voters indicated only their first rank, C would win (not B).

1.5.3 Dodgson's method

42 voters	26 voters	21 voters	11 voters
B	A	E	E
A	E	D	A
C	C	B	B
D	B	A	D
E	D	C	C

Table 1.5 Dodgson's method is vulnerable to no-show and twin paradoxes

In Table 1.5, A is closest to becoming the Condorcet winner, i.e. it is the Dodgson winner.[4]

[3] This subsection is partly based on the author's correspondence with Dan S. Felsenthal on May 25, 2001.

[4] This example is an adaptation of one given by Fishburn (1977, 478).

Now take 20 out the 21 voter group out. Then B becomes the Condorcet and, thus, Dodgson winner. B is preferred to A by the abstainers, demonstrating Dodgson's vulnerability to the no-show paradox. Adding those 20 "twins" back to retrieve the original profile shows that Dodgson is also vulnerable to the twin paradox.

1.5.4 Pareto violations, no-show and twin paradoxes of Schwartz

As will be recalled, the Pareto condition states: if everybody strictly prefers x to y, then y is not chosen. Schwartz's method violates this condition as shown in Table 1.6.

1 voter	1 voter	1 voter
A	D	B
B	C	D
D	A	C
C	B	A

Table 1.6 Schwartz's method violates the Pareto condition

Table 1.6 exhibits a top cycle: $A \succ B \succ D \succ C \succ A$. Hence this is the choice set of Schwartz. Yet, C is Pareto dominated by D.

To find out whether Schwartz is vulnerable to the no-show paradox we have to make assumptions regarding the risk-posture of voters. If they are assumed to be risk-averse, then the following example demonstrates the vulnerability of Schwartz to both no-show and twin paradoxes.

23 voters	28 voters	49 voters
A	B	C
B	C	A
C	A	B

Table 1.7 Schwartz's method is vulnerable to no-show and twin paradoxes if voters are risk-averse

In Table 1.7 the Schwartz choice set is A, B, C. With 4 voters from the BCA voters abstaining, C becomes the Condorcet – and thus Schwartz – winner. Starting from the 96-voter profile and adding BCA voters one by one, we can demonstrate the twin paradox.

In case of risk-neutral voters, we can demonstrate these paradoxes through the profile of Table 1.8:

23 voters	28 voters	49 voters
A	B	C
B	C	D
D	D	A
C	A	B

Table 1.8 Schwartz's method and risk-neutral voters

Here the Schwartz (GOCHA) choice set is A, B, C, D. With 4 voters of the BCDA group abstaining, C again becomes the Condorcet winner and is thus elected. This shows the no show paradox. The twin paradox emerges when one starts with the 96-voter profile and adds BCDA voters one by one as above.

1.5.5 Max-min rule

The max-min rule is also vulnerable to no show, truncation and twin paradoxes. Table 1.9 illustrating this is an adaptation of Pérez (1995).

5 voters	4 voters	3 voters	3 voters	4 voters
D	B	A	A	C
B	C	D	D	A
C	A	C	B	B
A	D	B	C	D

Table 1.9 Max-min method is vulnerable to no show, truncation and twin paradoxes

The outranking matrix of the Table 1.9 profile is in Table 1.10.

	A	B	C	D	row min
A	-	10	6	14	6
B	9	-	12	8	8
C	13	7	-	8	7
D	5	11	11	-	5

Table 1.10 Outranking matrix of Table 1.9

Thus, B is elected. However, with the 4 CABD voters abstaining, the outcome would be A. With only 1 CABD voter added to the 15-voter profile, A is still elected. If one then adds 3 "twins" of the CABD voter, one ends up with B being elected. Hence twins are not welcome. If those 4 voters reveal their first preference only, the minimum entry in B's row drops to 4 and C emerges as the winner. Hence the truncation paradox. This outcome assumes that winners are determined on the basis of minimum support in pairwise comparisons. If a voter does not reveal his preference between x and y, he gives no votes to either one in the corresponding pairwise comparison. This is in line with Brams (1982) who first introduced the notion of preference truncation. Of course, other interpretations can be envisaged.

1.5.6 Young fails on no show and twin paradoxes

Young's method is a Condorcet extension that looks for the largest subset of voters which contains a Condorcet winner and elects the Condorcet winner of that subset of voters. Being a Condorcet extension, Young's rule is also vulnerable to the no show and twin paradoxes as illustrated by Table 1.11. The illustration is again inspired by and adapted from Pérez (2001) and Moulin (1988):

11	10	10	2	2	2	1	1
B	E	A	E	E	C	D	A
A	C	C	C	D	B	C	B
D	B	D	D	C	A	B	D
E	D	B	B	B	D	A	E
C	A	E	A	A	E	E	C

Table 1.11 Young's method is vulnerable to no show and twin paradoxes

In this profile E is elected (needs only 12 removals). Add now 10 voters with ranking EDABC. This makes D the Condorcet winner. Hence, the 10 added voters are better off abstaining. Indeed we have an instance of the strong version of the no show paradox. Obviously, twins are not always welcome here.

1.5.7 Kemeny fails on no show and twin paradoxes

The example of subsection 1.5.5 is applicable here. In the 15-voter profile (the four left-most groups of voters), the Kemeny-ranking is DBCA. Now add

4 voters with DABC ranking. A now becomes the Condorcet and Kemeny winner. Hence these four voters are better off not voting.

The twin paradox occurs when we start with the 15-voter profile adding voters one by one until the winner changes from D to A. The last added voter is the unwelcome twin.

Counterexamples are, indeed, important in proving incompatibilities of systems and criteria. However, they vary a great deal in terms of the underlying difficulty of constructing them. The above counterexamples dealing with the no show paradox and Condorcet extension methods show that even though a general result – here due to Moulin and Pérez – is known, it is not necessarily straight-forward to find examples to illustrate the incompatibility. This suggests that perhaps the compatibility should be viewed as a matter of degree rather than a dichotomy. In fact, we are here encountering the same problem as when discussing the relevance of simulation models: how often are problematic profiles likely to emerge? We just don't know, but if the difficulty of finding examples of some incompatibilities – e.g. between Young's method and invulnerability to the no show paradox – is anything to go by, some of the problematic profiles occur only in very specific circumstances. Hence their practical relevance is limited.

In addition to the empirical frequency of problematic profiles, the relevance of choice theoretic results also hinges upon the acceptability of the assumptions made in the theory. This is an issue we now turn to.

1.6 Another look at behavioral assumptions

The bulk of social choice theory is based on the assumption that the individuals are endowed with complete and transitive preference relations over the alternatives. While there are good grounds for making this assumption, it is not difficult to construct examples where a reasonable individual might not satisfy it. Consider Table 1.12.

The Dictator of Universities (DU, so far a purely fictitious figure) ponders upon the evaluation of three universities A, B and C in terms of three criteria: (i) research output (scholarly publications), (ii) teaching output (degrees), (iii) external impact (expert assignments, media visibility, R & D projects, etc.). DU deems these criteria of roughly equal importance in determining the future funding of the universities. His observations are summarized in Table 1.12.

Since the criteria are of roughly equal importance DU comes up with the following list of binary preferences: $A \succ B \succ C \succ A \succ \ldots$. There is nothing unreasonable in this obviously intransitive preference relation. So, perhaps we should give some thought on alternative foundations of choice theory. There are basically two ways to proceed in searching for those foundations:

criterion (i)	criterion (ii)	criterion (iii)
A	B	C
B	C	A
C	A	B

Table 1.12 Performance of three universities on three criteria

(i) assume something less demanding, or (ii) something more demanding than preference rankings.

1.6.1 Asking for less than rankings

It is well-known that Arrow's focus on social welfare functions was eventually replaced by apparently less demanding concept of social choice function. In similar vein, one could replace the notion of complete and transitive individual preference relation with that of a choice function, i.e. a rule indicating for each subset of alternatives the set of best alternatives. In Arrovian spirit one could then look for plausible conditions on methods of aggregating the individual choice functions into collective ones.

The following would seem plausible conditions on collective choices based on individual choice functions:

- citizen sovereignty: for any alternative x, there exists a set of individual choice function values so that x will be elected,
- choice-set monotonicity: if x is elected under some profile of individual choices, then x should also be elected if more individuals include x in their individual choices
- neutrality
- anonymity
- choice-set Pareto: if all individuals include x in their individual choice sets, then the aggregation rule includes x as well , and if no voter includes y in their individual choice set, then y is not included in the collective choice.
- Chernoff's condition: if an alternative is among winners in a large set of alternatives, it should also be among the winners in every subset it belongs to (Chernoff's postulate 4, Chernoff 1954, 429).
- Concordance: suppose that the winners in two subsets of alternatives have some common alternatives. Then the rule is concordant if these common alternatives are also among the winners in the union of the two subsets (Chernoff's postulate 10; Chernoff 1954, 432; Aizerman and Aleskerov 1995, 19-20).

Incompatibilities can also be encountered in this less demanding setting. To wit, consider two rules for making collective choices. Rule 1: whenever an alternative is included in the choice sets of a majority of voters, it will be elected. Rule 2 (plurality): whichever alternative is included in more numerous choice sets than any other alternative, is elected. Table 1.13 presents an example of a three-member voting body pondering upon the choice from $\{x, y, z\}$. The individual choice sets as well as those resulting from the application of Rule 1 and Rule 2 are indicated (Aizerman and Aleskerov 1995, 237).

alt. set	ind. choice sets			Rule 1	Rule 2
	ind.1	ind. 2	ind. 3		
$\{x, y, z\}$	$\{x\}$	$\{z\}$	$\{y\}$	$\emptyset$	$\{x, y, z\}$
$\{x, y\}$	$\{x\}$	$\{x\}$	$\{y\}$	$\{x\}$	$\{x\}$
$\{x, z\}$	$\{x\}$	$\{z\}$	$\{x\}$	$\{x\}$	$\{x\}$
$\{y, z\}$	$\{y\}$	$\{z\}$	$\{y\}$	$\{y\}$	$\{y\}$

Table 1.13 Two choice function aggregation rules

Concordance is not satisfied by Rule 1, since x is chosen from $\{x, y\}$ and $\{x, z\}$, but not from $\{x, y, z\}$. Rule 2 fails on Chernoff since z is in the choice set from $\{x, y, z\}$, but from $\{x, z\}$. It is also worth noticing that plurality (Rule 2),but not majority (Rule 1) fails on choice-set monotonicity.

Aggregating choice profiles instead of preference ones is in a way natural when one is dealing with collective choices rather than rankings. Yet, as we just saw, incompatibilities between various desiderata can be encountered here as well. Individual choice functions are less demanding than preference rankings. All one needs to assume regarding the underlying preference relations is completeness. A step towards more demanding ways of expressing preferences is individual preference tournament. Tournaments – it will be recalled – are complete and asymmetric relations. One could argue that when the individuals take different properties or aspects of choice options into account when forming their preference between different pairs of options, the satisfaction of completeness and asymmetry comes naturally. Yet, transitivity is less obvious. Tversky's (1969) experiments with choices involving pairs of risky prospects illustrate this.

Now, if tournaments instead of rankings or choice functions are taken as proper descriptions of individual opinions, we have readily at hand several solution concepts, to wit, the uncovered set, top cycle set, Copeland winners, the Banks set (Banks 1985; Miller 1995; Moulin 1986). Typically these specify large subsets of alternatives as winners and are, thus, relatively unhelpful in settings where single winners are sought. There are basically two ways of utilizing individual tournament matrices in making collective decisions:

1. Given the individual $k \times k$ tournaments, construct the corresponding collective one of the same dimension by inserting 1 to position (i, j) if more than $n/2$ individuals have 1 in the (i, j) position. Otherwise, insert 0 to

this position. The row sums then indicate the Copeland scores. Rows with sum equal to zero correspond to the Condorcet losers, those with sums equal to $k - 1$ to the Condorcet winners. Uncovered and Bank's sets can be computed as well (the latter, though, is computationally hard). Also Dodgson scores can determined.

2. Construct the collective opinion matrix as an outranking matrix where the entry in the (i, j) position equals the number of individuals with 1 in the (i, j) position. The row sums then indicate the "Borda scores". Max-min scores can also be determined.

So, the concepts of preference aggregation can be re-invoked in tournament aggregation.

1.6.2 Asking for more than rankings

Another way of responding to social choice incompatibilities is to start from more, rather than less, demanding notions than individual preference rankings. In fact, this response has a firm foundation in the classic utility theory. Over the past decade it has been reiterated by several authors. To quote one them (Hillinger 2005):

> ... a new 'paradox of voting': It is theorists' fixation on a context dependent and ordinal preference scale; the most primitive scale imaginable and the mother of all paradoxes.

The step from complete and transitive preference relations to utility functions representing these functions is short, in fact, in the finite alternative sets nonexistent. Given the preference relations one can eo ipso construct the corresponding utility functions. These might then be used in preference aggregation. Since the cardinal utilities thereby obtained are unique up to affine transformations, one can transform all utility functions into the same scale by restricting the range of values assigned to each alternative. The utility values can, then, be used in defining social choice functions in many ways. Hillinger (2005) suggests the following. Let P_i be a strict preference relation of voter i and let P_i assign the set of candidates into disjoint subsets $A_1, \ldots, A_K, K \geq 1$ such that the voter is indifferent between candidates in the same subset and strictly prefers $a_i \in A_i$ to candidate $a_j \in A_j$ iff $i > j$. K is given independently of the number of candidates. For a given K, the voter is asked to assign to each candidate one of the numbers $x_0, x_0 + 1, \ldots, x_0 + K - 1$. The utilitarian voting winner is the alternative with the largest arithmetic mean or sum of assigned numbers.

This method simply sums up the scores – or utilities expressed in the $[x_0, x_0 + k - 1]$ interval – to determine the winning candidate or ranking of the candidates. Now this method has many names. Riker (1982) calls it Bentham's method, Hillinger the utilitarian or evaluative voting and Warren

D. Smith the range voting. It is worth pointing out that the cumulative voting method whereby each voter can freely allocate a fixed stock of votes to various candidates, is not equivalent to utilitarian voting, although somewhat similar in spirit to the latter.

The just mentioned methods invoke a new criterion of performance: the maximization of collective utility. What is then maximized is the sum of utilities assigned to an alternative by all voters. Summation is, of course, just one possible way of handling the utilities. In addition to various non-anonymous (weighted) methods of summation, one could also maximize the product of the utility values. Riker calls this Nash's method with an obvious reference to the Nash product in bargaining theory.

The most recent entrant in the class of systems dealt with in this subsection is the majoritarian judgment introduced and elaborated by Balinski and Laraki (2007). It works as follows:

1. each voter gives each candidate an ordinal grade (e.g. poor, medium, good, excellent)
2. the median grade of each candidate is determined
3. the winner is the candidate with the highest median grade
4. a specific tie-breaking rule is defined

Felsenthal and Machover (2008) have given an evaluation of the majoritarian judgment in terms of criteria applied in the ordinal social choice framework. The result is a typical mixture of good and bad showings. To summarize their evaluation: the majoritarian judgment does satisfy the Chernoff property, it is monotonic and is immune to cloning. These are undoubtedly desirable properties. In contrast to these, the system is inconsistent, vulnerable to the no-show paradox and may result in a Condorcet loser.

The evaluation shows that the ordinal choice theory criteria can be applied to voting systems that utilize richer information about voter opinions than just the ranking of candidates. However, one could ask whether the evaluation based solely on criteria borrowed from the ranking environment misses something relevant, viz. the fact that these systems are devised to attain goals (such as maximizing social welfare) that cannot be expressed in terms of ordinal concepts only. If this is the case, then at least some of the evaluation criteria should be specific to systems based on aggregating cardinal utilities. For example a person resorting to utilitarian voting might not be at all worried if the method fails on Condorcet winner criterion as long as it maximizes the sum of expressed utilities. Much work remains to be done in devising non-trivial criteria for such more specific evaluations. Until they have been invented, the best we can do is to proceed in the manner suggested by Felsenthal and Machover (2008).

1.7 Concluding remarks

The most significant results of social choice theory pertain to compatibilities of various choice desiderata. Some of these take the form of proving the incompatibility of various properties of choice rules, others do the same for specific choice rules and voting procedures. The choice of the best rule is complicated by the sheer number of desiderata that one intuitively would like to see fulfilled, but even within relatively small subsets of important choice criteria one typically finds no procedure that would satisfy them all. Even dominance relations between procedures are uncommon. Since the procedures are intended for use in future collective decision making contexts, their success in avoiding anomalies of paradoxes is highly contingent upon encountering problematic preference profiles. Probability models and simulations have often been resorted to in order to obtain estimates about the theoretical frequencies of problematic profiles. This approach can be complemented by another one focusing on the difficulty of finding counterexamples showing various incompatibilities. Arguably it is only by looking at the structure or details of the problematic profiles that one can obtain information about their likelihood in practice. In the preceding we have also briefly touched upon alternative foundations of choice theory. Some of them require more information from the individuals, others less than the ordinal ranking approach. Setting up useful criteria for analyzing systems aggregating this new type of information is still largely to be done.

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